Pairwise comparisons and rankings: mathematical aspects from gauge theory to topology

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Pairwise comparisons and rankings: mathematical aspects from gauge theory to topology

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Presentation of the talk

1. Weights as discretized gauges, etc.
    - Yang-Mills theory in physics? in loop quantum gravity?
    - Analogy with inconsistency reduction
2. Mimization problem and gradient method
    - Necessary framework for a gradient method
    - inconsistency indicators and the volatility of rankings
    - A "prehistoric" gradient method; the orthogonal consistencization
3. The problem of equal ranking
    - equal ranking, equal weights and consistent pairwise comparisons
    - The characteristic ranking matrix
    - strict ranking as a minimization problem

The gauge problem in physics

A principal bundle P with structure group G over a manifold M is defined by:

- a free action on the right of G on P,
- a smooth surjective map $\pi : P \to M$ such that $\forall x \in M, \pi^{-1}(x)$ is an orbit of G.

A **gauge** on P is a smooth section $M \to P$. It identifies P with $M \times G$ (if it exists).
Various gauges in physics:

- physically motivated: temperature (gauges: Kelvin, Celcius ...), altitude, Galilean frames
- more discutable gauges: axial gauge, Coulomb gauge for weak/strong interactions

How to build a gauge by connections

A connection lifts paths on the base M to horizontal paths on P. The defect of a loop to be a horizontal loop is called holonomy, and the curvature of a connection is an infinitesimal holonomy. Read on a simplex, connexions on a trivial principal bundle with abelian structure group read as a map a from edges to G, and the holonomy of the loop $i \to j \to k \to i$ is

$$a_{i,j} a_{j,k} a_{k,i}.$$

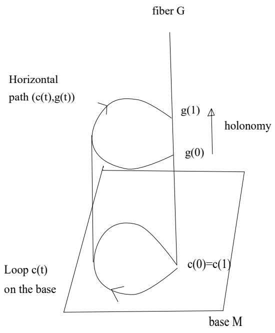

Figure: Holonomy of a non-flat connection.

The correspondence with pairwise comparisons

discrete Yang-Mills formalism	Pairwise Comparisons (PC)
connection	PC matrix
flat connection	consistent PC matrix
curvature = loop holonomy	inconsistency
classical Yang-Mills functional	quadratic average of inconsistency on triads
"sup" Yang-Mills functional	Koczkodaj's inconsistency indicator

Necessary framework for the gradient method

- a convex domain $\Omega \subset \mathbb{R}^n$
- equipped with a Riemannian metric g, that is a smooth map from Ω to symetric definite bilinear forms
- a functional, (ideally a strictly convex functional i.e. $\forall x, y$, $f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2}$) differentiable everywhere.

Theorem

The integral curves of the gradient vector field $-\nabla f = -g^{-1}(df)$ are the shortest paths for the metric g that link a point in Ω with a point realizing a (the) minimum of f.

The gradient method applied to a class of inconsistency indicators

[M,Mazurek, Cernanova, 2023] the sample family of inconsistency indicators are of the type

$$ii_n^p = ||(|a_{i,j}a_{j,i}a_{k,i} - 1|)_{1 \leq i < j < k \leq n}||_p$$

where

$$||X||_p = \left(\sum |X_l|^p\right)^{1/p}, \quad p \in \mathbb{R}^* \cup \{+\infty\}.$$

Then

- the discretized gradient methods indeed produce approximately consistent PC matrices
- two inconsistency indicators in this family may lead, to the same PC matrix, to non-equivalent rankings.

$\Rightarrow$ the inconsistency indicator may be used to modelize the requirements of the decision maker. more largely than in encoding approximate consistency.

A prehistoric gradient method

"In an Euclidean space, the shortest path between two points is a segment".
Moreover,
"In an Euclidean space, there exists an unique orthogonal projection of a point into a convex set"
These two principles are applied by Koczkodaj and al. to project PC_n to CPC_n in additive pairwise comparisons, with respect to any scalar product on PC_n.

Claim: The gradient method for ii_n^2, expressed additively, and the canonical scalar product on $\mathbb{R}^{n(n-1)/2}$ corresponds to this method.

Strict rankings and PC matrices

Last section is based on [M, preprint 2024]. A *strict ranking* is defined by a family of weights $(w_i)_{i \in \mathbb{N}_n}$ that define an injective map $\mathbb{N}_n \to \mathbb{R}_+^*$,

$\Leftrightarrow$

$\exists ! \sigma \in \mathfrak{S}_n$ such that

$$w_{\sigma(1)} < \cdots < w_{\sigma(n)}$$

$\Leftrightarrow$

$$i = j \quad \Leftrightarrow \quad a_{i,j} = \frac{w_j}{w_i} = 1.$$

The $\mathcal{R}-$condition on PC_n

$$\mathcal{R}PC_n = \left\{ (a_{i,j})_{(i,j)\in\mathbb{N}_n^2} \in PC_n \,|\, i = j \Leftrightarrow a_{i,j} = 1 \right\}$$

with connected components characterized by the set of indexes

$$I((a_{i,j})_{(i,j)\in\mathbb{N}_n^2}) = \left\{ (i,j) \in \mathbb{N}_n^2 \,|\, i < j \text{ and } a_{i,j} > 1 \right\}.$$

locus $=$ one connected component
admissible locus $= \exists \sigma \in \mathfrak{S}_n, \forall i < j, a_{\sigma(i),\sigma(j)} < 1.$

Theorem

If $A \in \mathcal{R}PC_n$ is in an admissible locus, the permutation σ is unique and it defines a reordering (a ranking) in $\mathbb{N}_n$ compatible with any family of weights generated by any consistent PC matrix in the same admissible locus.

Existence of non admissible locus

Given $(a_{i,j})_{(i,j)\in\mathbb{N}_n^2} \in PC_n$ we associate its characteristic ranking matrix with coefficients

$$c_{i,j} = \mathrm{sign}(\log(a_{i,j})) = \begin{cases} 0 & \text{if} \quad a_{i,j} = 1 \\ 1 & \text{if} \quad a_{i,j} > 1 \\ -1 & \text{if} \quad 0 < a_{i,j} < 1 \end{cases} .$$

Theorem

$\forall n \geq 3, \mathcal{R}PC_n$ has non admissible loci.

Example

For $n = 3$, the 2 non-admissible have characteristic ranking matrices $\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$

Strict ranking and consistency: a dream?

Theorem

The gradient method for some inconsistency indexes tested in [MMC2023] preserve neither admissible loci nor even loci.

Theorem

Orthogonal consistencization do not preserve the $\mathcal{R}-$condition.

Remark

The existence of non-admissible loci make impossible the existence of an inconsistency index for which the preserves the loci in $\mathcal{R}PC_n$.

A minimization problem for strict ranking and consistency

$$\mathcal{A}PC_n = PC_n - \overline{\mathcal{R}}CPC_n. \text{ Let } \Phi : \mathcal{A}PC_n \to \mathbb{R}$$

defined by $\Phi((a_{i,j})_{(i,j)^i n \mathbb{N}_n^2}) =$

$$\left(\prod_{(i,j)\in\mathbb{N}_n^2, i<j} \frac{ii(A)}{(\log(a_{i,j}))^2 + (ii(A))^{n^2/2}} \right) \sum_{(i,j)\in\mathbb{N}_n^2, i<j} ((\log(a_{i,j}))^4 + 1).$$

Theorem

It is a functional

- *which is $\mathbb{R}_+ -$valued,*
- *which vanishes only on consistent PC matrices that satisfy the $\mathcal{R}-$condition*

Thank you for your attention!

Main references:

- Magnot, J-P.; preprint 2024, still "on hold" in arXiv.
- Ellingsen, A.; Magnot, J-P.; Lundholm, D.; (2024)
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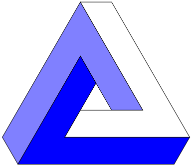

Figure: Penrose tribar.